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Nonlinear resonant wave motion of a radiating gas

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A method due to Chester is applied to the theoretical study of resonant wave motion of a radiatively active gas. The inviscid non-conducting grey gas is confined between two infinite parallel walls, and a one-dimensional wave motion is driven by a sinusoidally varying input of black-body radiation from one of the walls. For sufficiently weak driving radiation, the motion can be described by the general solution of the classical wave equation (with the functional form of the solution still undetermined) plus a particular solution due to the driving radiation. When the driving is done at or near a resonant frequency, however, nonlinearities and perturbations in spontaneous emission from the gas must be taken into account before application of the boundary conditions. Such application then leads to a nonlinear integral equation governing the undetermined function in the general solution of the wave equation. This equation is solved numerically by the method of parametric differentiation.

In a frequency range around resonance and for a sufficiently weak relative level of spontaneous emission, the nonlinearities give rise to shock waves (numbering N at the Nth resonant frequency) that are repeatedly reflected at the walls. The perturbations in spontaneous emission give rise to damping, however, and for sufficiently high levels of emission the shock waves disappear. Specific results for various values of optical thickness and various relative levels of spontaneous emission are presented at frequencies in a range around the first resonance.

1. Introduction

This paper studies the interaction of gasdynamic wave motion and the radiative transfer of energy. We are concerned in particular with a resonant phenomenon that can occur when an absorbing and emitting gas, driven by a sinusoidally varying input of radiation, is confined between two infinite parallel walls. As illustrated in figure 1 (*a*), one wall is taken to be perfectly reflecting, while the other is black with a temperature that varies sinusoidally around the mean temperature of the system. The radiative energy flux emitted from the black wall is absorbed by the gas, and the resulting pressure fluctuations drive a one-dimensional wave motion perpendicular to the walls. This situation is intended to be representative, in so far as radiative effects are concerned, of the experimental device shown in figure 1 (*b*). To isolate the radiative effects, we dis-

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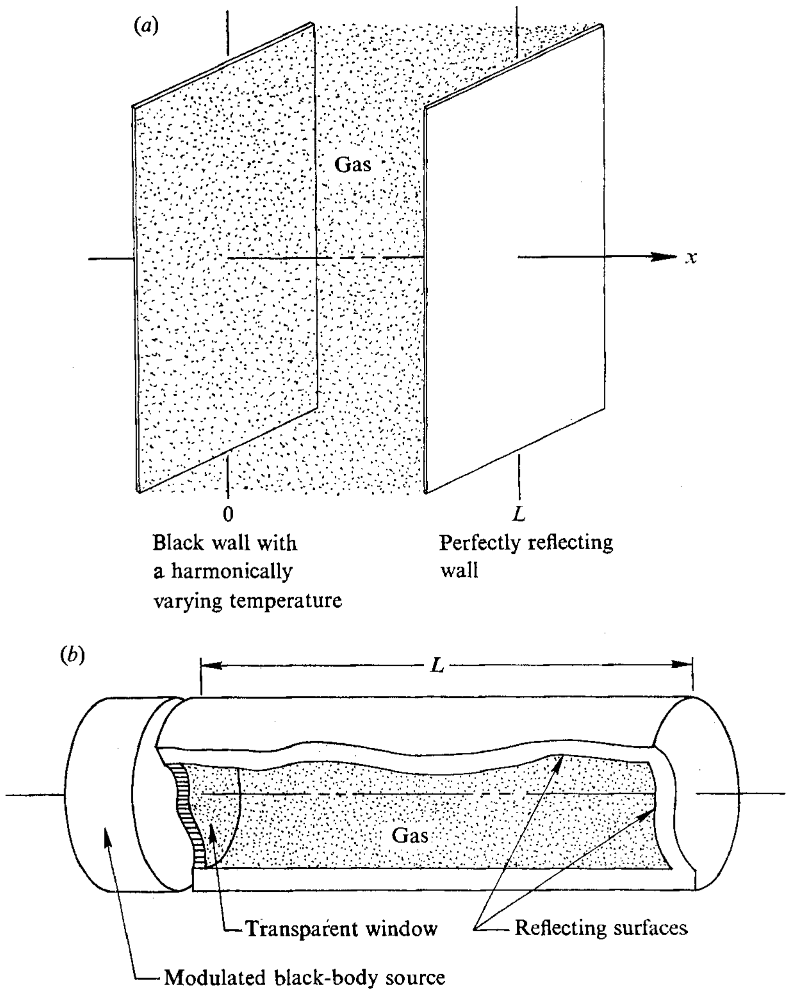

FIGURE 1. (a) Analytical model. (b) Experimental device.

regard all influences of molecular transport, notably those that arise from the boundary layer on the cylindrical side wall.

The resonant phenomenon with which we are concerned occurs when the wave motion is driven at a frequency close to one of the natural acoustic frequencies of the system. As with resonant phenomena generally, the response at resonance is markedly larger than that away from resonance. Furthermore, effects that are small enough to be neglected in the non-resonant situation play essential roles in governing the wave motion near resonance. In the present problem it will be seen that these effects are (i) perturbations in spontaneous emission of radiation arising from the temperature fluctuations in the gas, and (ii) gasdynamic nonlinearities. In fact, the simplest conceivable theory, which neglects both of these effects, predicts the impossible result of an infinite response at resonance.

In an earlier analysis of the problem, Long & Vincenti (1967) retained the perturbations in spontaneous emission but neglected nonlinearities. Their results show that, for conditions attainable in the laboratory, the amplitude of the pressure variation on the reflecting wall, though prevented from becoming infinite by the spontaneous emission, could still be as much as three orders of magnitude greater at resonance than away from resonance.

Because of the large resonant response given by the linearized theory, one wonders whether gasdynamic nonlinearities may not also be significant. A comprehensive theoretical treatment of nonlinear resonant wave motion has been carried out by Chester (1964). In his case, a non-radiating gas in a closed tube is driven by a small amplitude sinusoidal piston motion of one of the end walls. Chester's analysis shows that, near resonance, gasdynamic nonlinearities give rise to shock waves that travel back and forth in the tube, being repeatedly reflected at the end walls. Near the first resonant frequency one such wave exists; near the Nth resonant frequency N waves appear. Because of the similarity of the two problems, one expects that the nonlinear effects that appear in the piston-driven problem may also be important in the radiatively driven problem.

For his treatment of the piston-driven problem, Chester devised an ingenious method of analysis; this method can be employed to solve the radiatively driven problem as well. Chester's method allows inclusion not only of nonlinearities but also of other effects that play an important role near resonance. In his piston-driven case Chester included the effect of longitudinal viscosity and thermal conductivity and of the boundary layer on the side wall of the tube. For our radiatively driven problem in the assumed absence of molecular-transport effects, we must include the perturbations in spontaneous emission.

The key to the application of Chester's method lies in an understanding of why a simplified linear theory fails at resonance. Consider a situation in which the transport of energy by wave motion is much larger than that arising from perturbations in spontaneous emission. (This is, in fact, the situation for the experimental device.) Under these conditions the simplest conceivable theory, already mentioned, is a linearized theory in which the perturbations in spontaneous emission are neglected. The equation that governs the velocity of the gas is then an inhomogeneous wave equation with the isentropic speed of sound as the wave speed. The inhomogeneity represents the effect of the driving radiation. The particular solution u_p of the equation is selected so that it vanishes at the black wall, which is at $x = 0$. The complementary function is the general solution of the homogeneous wave equation, i.e.

$$u_c = f_+(t - x/a_{s0}) + f_-(t + x/a_{s0}),$$

where a_{s0} is the isentropic speed of sound evaluated at the mean state of the gas. The complementary function can be made to vanish at $x = 0$ by taking

$$f_+(t) = -f_-(t) = f(t),$$

say. Since both u_p and u_c now vanish at $x = 0$, the boundary condition $u = 0$ is satisfied at the black wall. The function f is then determined by the condition that the complementary function must identically cancel the particular solution at $x = L$, thus satisfying the boundary condition at the reflecting wall.

This procedure gives acceptable results except when the system is near resonance. To understand why it then fails, consider some properties of the complementary function. It may be assumed that, after the system has been operating for a long time, the response will be periodic with the same period as the driving radiation. If the radian frequency of the driving radiation is ω, f is thus periodic in time with period $2\pi/\omega$. As a consequence, the complementary function $u_c = f(t-x/a_{s0}) - f(t+x/a_{s0})$ will vanish wherever $x = N\pi a_{s0}/\omega$, where N is an integer. When the position of the reflecting wall coincides with one of these nodes, that is, when $L = N\pi a_{s0}/\omega$, the system is at resonance. Since the complementary function then vanishes at the reflecting wall, however, it cannot cancel the particular solution at that point. In attempting to satisfy the boundary condition, this theory can only predict an infinite amplitude for the complementary function, and hence for the gasdynamic response. Slightly off the resonant frequency, the response, though still finite, is so large that the small perturbation assumption of the linear theory is violated.

If the radiative driving of the gas is sufficiently weak, which is the case in the laboratory situation, one expects that the response will in fact remain small even at resonance. The main part of the equation governing the wave motion should then still be the same as the linear wave equation of the simplified theory. Near resonance, however, the equation must be made more realistic before one applies the boundary condition in the neighbourhood of a node of the complementary function. Chester suggests that this be done by iteration as follows. The nonlinear terms and any other ordinarily small terms that play an important role near resonance are first evaluated formally on the basis of the solution obtained from the linear wave equation. (In the present problem the only other such terms are those due to the perturbations in spontaneous emission.) Since it is the application of the boundary condition at $x = L$ that causes the solution of the linear wave equation to become infinite, one must use for this evaluation the solution in the still functionally undetermined form that is obtained *prior to* the application of the boundary condition. For this purpose it is sufficient to use only the complementary function $u_c = f(t-x/a_{s0}) - f(t+x/a_{s0})$, because near resonance this will predominate over u_p. The result of this iterative process is an inhomogeneous wave equation in which the inhomogeneous terms contain the still undetermined function f. Once this equation has been solved for the velocity in terms of f, the boundary conditions can then be applied at the reflecting wall. This results in a nonlinear integral equation, for the form of f, that is uniformly valid at resonance as well as away from resonance.

The validity of Chester's method is well established. The results of the original application to piston-driven wave motion of a non-radiating gas are in good agreement with several experimental investigations of the problem (for example, Cruikshank 1969), as well as with other theoretical investigations (Betchov 1958; Collins 1971; Mortell 1971). In a second application of his method, Chester (1968) treated the problem of resonant surface waves in a liquid in a rectangular tank. Here nonlinear effects, together with dissipation and dispersion, are significant. To test the theoretical results, Chester & Bones (1968) investigated the problem experimentally. Although the wave motion presented some novel features, they found that the theory adequately describes the phenomena.

The linearized theory of radiatively driven waves by Long & Vincenti, already cited, was followed by experimental investigations by Compton (1969) and Chapman (1970). Both these investigators used a device like that depicted in figure 1(*b*), with carbon dioxide as the radiatively active gas. Compton and Chapman did not observe shock waves when the system was at resonance. Evidently, the radiatively driven waves, being much weaker than piston-driven waves, were damped by the side-wall boundary layer to such an extent that nonlinear effects were unimportant even at resonance. This, as well as theoretical estimates made by Eninger (1971), suggests that the present nonlinear inviscid theory would be difficult to test experimentally. It is of interest, nevertheless, as one of the few theoretical studies of the interaction of gasdynamic nonlinearities and radiative transfer that is not essentially numerical.

The main body of the paper is divided into four parts. In § 2 the problem is stated in mathematical terms, and equations appropriate for the application of Chester's method are derived. To this end several idealizations and approximations must be introduced, the most significant being those concerning the radiative transfer. In particular, the absorption coefficient is assumed to be independent of spectral frequency (grey-gas approximation), and the integro-exponential functions appearing in the equation of radiative heat addition are approximated by exponential functions. In § 3 Chester's method is discussed and applied, leading to the nonlinear integral equation governing f. The solution of this equation is taken up in § 4. Here, following analytical discussion of two special cases, the method of parametric differentiation is used to reduce the general nonlinear equation to linear integral equations, and the trapezoidal integration formula is applied to reduce these equations to a set of linear algebraic equations. These are solved on a computer. The resulting solutions are presented and discussed in § 5. These show how the response varies with radiative level, optical thickness and deviation from resonance. They also show that shock waves exist only below a certain critical radiative level, which decreases with increasing deviation from resonance.

2. Basic equations

The basic assumptions of the analysis are as follows. First, the gas is taken to be thermally and calorically perfect, inviscid and in local thermodynamic equilibrium. Second, the speed of light is assumed to be so large that it may be approximated as infinite. As a consequence there are no radiative contributions to the pressure or internal energy of the gas, and the temporal term in the equation of radiative transfer is negligible. Third, the radiative absorption coefficient is idealized as being independent of spectral frequency (grey-gas approximation). Further, since the local thermodynamic state of the gas is to deviate by only a small amount from the mean state, the resulting perturbations in the absorption coefficient will be small. Hence, in calculation of the first-order effect of radiative transfer, the absorption coefficient is assumed to be independent of perturbations in the state of the gas.

The problem is formulated in dimensionless form by introduction of the following dimensionless variables:

$$\bar{u}' = u/a_{s0}, \quad \bar{\rho} = \rho/\rho_0, \quad \bar{p} = p/p_0, \quad \bar{T} = T/T_0,$$
$$\bar{s}' = (s-s_0)/c_v, \quad \bar{q}^R = q^R/\sigma T_0^4, \quad \bar{x} = x\omega/a_{s0}, \quad \bar{t} = \omega t.$$

The quantities u, ρ, p, T, s and a_s are respectively the velocity, density, pressure, temperature, entropy and isentropic speed of sound of the gas. The subscript zero refers to the undisturbed, mean state of the gas, and a prime denotes a perturbation quantity. The quantity ω is the radian frequency of the sinusoidal temperature variation of the black wall. The radiative heat flux is denoted by q^R, σ is the Stefan–Boltzmann constant and c_v is the specific heat at constant volume. The variables x and t are respectively the distance from the black wall and the time.

The mathematical statement of the problem can then be written as follows.

Continuity equation:

$$\frac{\partial \bar{\rho}}{\partial \bar{t}} + \bar{\rho}\frac{\partial \bar{u}'}{\partial \bar{x}} + \bar{u}'\frac{\partial \bar{\rho}}{\partial \bar{x}} = 0. \tag{1}$$

Momentum equation:

$$\frac{\partial \bar{u}'}{\partial \bar{t}} + \bar{u}'\frac{\partial \bar{u}'}{\partial \bar{x}} + \frac{1}{\gamma \bar{\rho}}\frac{\partial \bar{p}}{\partial \bar{x}} = 0. \tag{2}$$

Energy equation:

$$\bar{\rho}\bar{T}\left(\frac{\partial \bar{s}'}{\partial \bar{t}} + \bar{u}'\frac{\partial \bar{s}'}{\partial \bar{x}}\right) = -\frac{\gamma}{Bo}\frac{\partial \bar{q}^R}{\partial \bar{x}}. \tag{3}$$

Equation of radiative heat addition:

$$-\frac{\partial \bar{q}^R}{\partial \bar{x}} = 2Bu\left\{(1+|\bar{T}_w'|\sin \bar{t})^4\{E_2(Bu\bar{x}) + E_2[Bu(2\bar{L}-\bar{x})]\}\right.$$
$$\left. + Bu\int_0^{\bar{L}} \bar{T}^4\{E_1(Bu|\bar{x}-\tilde{x}|) + E_1[Bu(2\bar{L}-\bar{x}-\tilde{x})]\}\,d\tilde{x} - 2\bar{T}^4\right\}. \tag{4}$$

Equations of state:

$$\bar{p} = \bar{\rho}\bar{T}, \quad \bar{p} = \bar{\rho}^\gamma \exp \bar{s}'. \tag{5}$$

Boundary conditions:

$$\bar{u}'(0,\bar{t}) = \bar{u}'(\bar{L},\bar{t}) = 0. \tag{6}$$

The functions $E_1(z)$ and $E_2(z)$ appearing in (4) are integro-exponential functions defined by

$$E_n(z) = \int_0^1 l^{n-2}\exp(-z/l)\,dl. \tag{7}$$

The dimensionless parameters appearing in the above statements are (i) the dimensionless distance $\bar{L}$ between the walls, (ii) the Bouguer number

$$Bu \equiv \alpha_0 a_{s0}/\omega,$$

where α_0 is the grey-gas absorption coefficient, (iii) the Boltzmann number

$$Bo \equiv c_p \rho_0 a_{s0}/\sigma T_0^3,$$

where c_p is the specific heat at constant pressure, (iv) the dimensionless amplitude $|\bar{T}_w'|$ of the sinusoidal temperature variation of the black wall and (v) the ratio of specific heats $\gamma \equiv c_p/c_v$.

The parameter $\bar{L}$ is the ratio of the distance L between the walls to the characteristic length a_{s0}/ω for the wave motion. Since resonance occurs whenever $L = N\pi a_{s0}/\omega$, the dimensionless condition for resonance is simply $\bar{L} = N\pi$.

The Bouguer number Bu is the ratio of the characteristic length a_{s0}/ω for the wave motion to the mean free path $1/\alpha_0$ for photons. The situation is optically thick when the ratio of the distance between the walls to the mean free path of photons is large, that is, $L\alpha_0 \gg 1$ or alternatively $\bar{L}Bu \gg 1$. When $\bar{L}Bu \gg 1$, the situation is optically thin.

The Boltzmann number Bo is a measure of the importance of transport of energy by wave motion relative to transfer of energy by radiation. In the laboratory device the value of Bo is large. Long & Vincenti (1967), for example, estimate values in the range from 500 to 10 000. A large value of Bo will be one of the basic approximations in the present theory.

The right-hand side of the equation of radiative heat addition (4) (cf. Vincenti & Kruger 1965) is made up of three contributions. The first term within the braces represents heat added to an element of gas at $\bar{x}$ as the result of radiation emitted by the black wall. The integral term represents heat added to the same element as the result of spontaneous emission of radiation from other elements of the gas. The final term gives the heat lost as the result of spontaneous emission from the element itself. The integro-exponential functions arise from the attenuation of the radiative energy as it passes through the absorbing gas. For example, in the integral term, $E_1(Bu|\bar{x}-\tilde{\bar{x}}|)$ comes from attenuation of the radiation that arrives at $\bar{x}$ directly from $\tilde{\bar{x}}$; $E_1[Bu(2\bar{L}-\bar{x}-\tilde{\bar{x}})]$ is from attenuation of the radiation that arrives at $\bar{x}$ from $\tilde{\bar{x}}$ by way of reflexion from the wall at $\bar{L}$.

We first derive equations appropriate for the application of Chester's method. The derivation for a non-radiating gas, but with the required nonlinear effects, is given by Lighthill (1956) and Chester (1964). The presentation here follows these derivations with some modification to provide for radiative transfer.

A characteristic of resonant systems generally is that the resonant response is, in some sense, much larger than the driving mechanism. We can take advantage of this in the present situation to neglect certain unimportant terms. First consider the driving radiation. It is coupled to the gasdynamics through the energy equation (3). Since the nonlinear convection $\bar{u}'\,\partial\bar{s}'/\partial\bar{x}$ of entropy is much smaller than $\partial\bar{s}'/\partial t$ and since $\bar{\rho}\bar{T} \approx 1$, we see from equation (3) that the entropy perturbation $\bar{s}'$ and the driving radiation are of the same order, say $\bar{S}'$. The gasdynamic response, on the other hand, is characterized by the perturbations in pressure, density, temperature and velocity. These are all of the same order, say $\bar{U}'$. As a consequence of the resonant response being much greater than the driving mechanism, we thus have the near-resonance condition $\bar{U}' \gg \bar{S}'$. Since it is necessary to keep only the most significant nonlinear terms, those of order $\bar{U}'^2$ will be retained in the derivation, while those of order $\bar{U}'\bar{S}'$, $\bar{S}'^2$, or smaller will be neglected.

To begin with, the pressure gradient is written as

$$\frac{\partial \bar{p}}{\partial \bar{x}} = \left.\frac{\partial \bar{p}}{\partial \bar{\rho}}\right|_{\bar{s}'} \frac{\partial \bar{\rho}}{\partial \bar{x}} + \left.\frac{\partial \bar{p}}{\partial \bar{s}'}\right|_{\bar{\rho}} \frac{\partial \bar{s}'}{\partial \bar{x}}. \tag{8}$$

With the aid of (5) this can be written as

$$\frac{\partial \bar{p}}{\partial \bar{x}} = \gamma \bar{c}^2 \exp \bar{s}' \frac{\partial \bar{\rho}}{\partial \bar{x}} + \bar{\rho}\bar{c}^2 \exp \bar{s}' \frac{\partial \bar{s}'}{\partial \bar{x}}, \tag{9}$$

where $\bar{c}$ is given by

$$\bar{c}^2 = \bar{\rho}^{\gamma-1}. \tag{10}$$

Equation (9) is used to eliminate the pressure from the momentum equation (2). Next, the density is eliminated from both the momentum equation and the continuity equation (1) through introduction of the perturbation quantity

$$\bar{c}' = \bar{\rho}^{\frac{1}{2}(\gamma-1)} - 1. \tag{11}$$

This quantity, being of the same order as perturbations in the density, is also of order $\bar{U}'$. The momentum equation, in terms of $\bar{u}'$, $\bar{c}'$ and $\bar{s}'$, is added to and subtracted from the continuity equation, in terms of $\bar{u}'$ and $\bar{c}'$. The resulting equations are

$$\left(\frac{\partial}{\partial \bar{t}} \pm \frac{\partial}{\partial \bar{x}}\right)\left(\bar{u}' \pm \frac{2\bar{c}'}{\gamma-1}\right) = -(\bar{u}' \pm \bar{c}')\frac{\partial}{\partial \bar{x}}\left(\bar{u}' \pm \frac{2\bar{c}'}{\gamma-1}\right) - \frac{1}{\gamma}\frac{\partial \bar{s}'}{\partial \bar{x}}. \tag{12}$$

In these equations, terms of order $\bar{U}'$, $\bar{U}'^2$ and $\bar{S}'$ are retained while terms of order $\bar{U}'\bar{S}'$ and smaller are neglected.

It is next necessary to calculate the entropy gradient in (12) by means of the energy equation (3) and the equation of radiative heat addition (4). By neglecting the term representing the convection of entropy and terms containing perturbations in the density and temperature, all of which are of order $\bar{U}'\bar{S}'$ or smaller, we write the energy equation as

$$\frac{\partial \bar{s}'}{\partial \bar{t}} = -\frac{\gamma}{Bo}\frac{\partial \bar{q}^R}{\partial \bar{x}}. \tag{13}$$

This equation is differentiated with respect to $\bar{x}$ and integrated with respect to $\bar{t}$ to give

$$\frac{\partial \bar{s}'}{\partial \bar{x}} = -\frac{\gamma}{Bo}\int_{-\infty}^{\bar{t}} \frac{\partial^2 \bar{q}^R}{\partial \bar{x}^2}\, d\tilde{\bar{t}}. \tag{14}$$

The equation of radiative heat addition (4) will be simplified by approximating the integro-exponential functions by pure exponentials as follows:

$$E_2(z) \approx a \exp(-bz) \quad \text{and} \quad E_1(z) \approx ab \exp(-bz).$$

This is the often-used 'substitute-kernel' or 'exponential' approximation. The constants a and b are chosen to obtain reasonable agreement between the integro-exponential functions and their exponential approximations. (For discussion of this approximation, see Vincenti & Kruger (1965, p. 483).) With the use of the exponential approximation and the introduction of the temperature perturbation $\bar{T}' = \bar{T} - 1$, equation (4) can be written to first order in $\bar{T}'$ as

$$-\frac{\partial \bar{q}^R}{\partial \bar{x}} = 8aBu\Big\{|\bar{T}'_w| \sin \bar{t}\{\exp(-bBu\bar{x}) + \exp[-bBu(2\bar{L} - x)]\}$$
$$+ bBu\int_0^{\bar{L}} \bar{T}'\{\exp(-Bu|\bar{x} - \tilde{\bar{x}}|) + \exp[-bBu(2\bar{L} - \bar{x} - \tilde{\bar{x}})]\}\, d\tilde{\bar{x}} - 2\bar{T}'/a\Big\}. \tag{15}$$

We now want to eliminate $\bar{T}'$ from this equation in favour of $\bar{c}'$. From (11) and (5), it is readily shown that $\bar{T}' = 2\bar{c}' + O(\bar{U}'^2) + O(\bar{S}')$. Since $\bar{c}'$ is of order $\bar{U}'$, near resonance, where $\bar{S}' \ll \bar{U}'$, the approximation $\bar{T}' \approx 2\bar{c}'$ is permissible. It may seem inconsistent to neglect terms here that are of order $\bar{U}'^2$ and $\bar{S}'$, since terms of these orders have already been retained in (12). The approximation is used, however, only in terms describing the radiative transfer of energy, and these will be multiplied in (14) by the small parameter $1/Bo$. Thus in making the approximation we are, in fact, neglecting terms of order $(1/Bo)\,\bar{U}'^2$ and $(1/Bo)\,\bar{S}'$.

Replacing $\bar{T}'$ by $2\bar{c}'$, we now use (15) to evaluate the right-hand side of (14). The resulting expression for $\partial\bar{s}'/\partial\bar{x}$ is then substituted into (12) to obtain

$$\left(\frac{\partial}{\partial\bar{t}} \pm \frac{\partial}{\partial\bar{x}}\right)\left(\bar{u}' \pm \frac{2\bar{c}'}{\gamma-1}\right) = -(\bar{u}' \pm \bar{c}')\frac{\partial}{\partial\bar{x}}\left(\bar{u}' \pm \frac{2\bar{c}'}{\gamma-1}\right)$$
$$-8ab\frac{Bu^2}{Bo}\Bigg\{|\bar{T}'_w|\cos\bar{t}\,\{\exp(-bBu\bar{x}) - \exp[-bBu(2\bar{L}-\bar{x})]\}$$
$$-2bBu\int_0^{\bar{L}}\hat{c}'\{\exp(-bBu|\bar{x}-\tilde{x}|)\,\mathrm{sgn}(\bar{x}-\tilde{x}) - \exp[-bBu(2\bar{L}-\bar{x}-\tilde{x})]\}\,d\tilde{x}$$
$$-\frac{4}{abBu}\frac{\partial\hat{c}'}{\partial\bar{x}}\Bigg\}. \qquad (16)$$

Here the notation

$$\hat{c}(\bar{x},\bar{t}) \equiv \int_{-\infty} \bar{c}'(\bar{x},\tilde{t})\,d\tilde{t}$$

has been introduced. The function sgn (z) in the integral term is $+1$ if z is positive and -1 if z is negative.

Equations (16) are two nonlinear integro-differential equations for $\bar{u}'$ and $\bar{c}'$. The terms on the left of the equals sign are the linear terms that govern ordinary acoustic motion. The first terms on the right represent the nonlinear effects of the gasdynamics. The remaining terms are due to radiative energy transfer. Note that these terms are linear. The first term inside the braces results from the driving radiation emitted by the black wall. The integral term and the last term are due to perturbations in spontaneous emission from the gas. The terms retained in (16) are of order $\bar{U}'$, $\bar{U}'^2$, $\bar{S}'$ or $(1/Bo)\,\bar{U}'$; consistent with the near-resonance approximation $\bar{U}' \gg \bar{S}'$ and the assumption $1/Bo \ll 1$, the terms neglected are of order $\bar{U}'\bar{S}'$, $(1/Bo)\,\bar{U}'^2$, $(1/Bo)\,\bar{S}'$ and smaller.

3. Application of Chester's method

As was discussed in the introduction, the key to Chester's method lies in understanding why a simplified linear theory fails near resonance. The equations appropriate for this simplified theory are equations (16) with the nonlinear terms and the terms involving perturbations in spontaneous emission deleted, that is,

$$\left(\frac{\partial}{\partial\bar{t}} \pm \frac{\partial}{\partial\bar{x}}\right)\left(\bar{u}' \pm \frac{2\bar{c}'}{\gamma-1}\right) = -8ab\frac{Bu^2}{Bo}\,|\bar{T}'_w|\cos\bar{t}\,\{\exp(-bBu\bar{x}) - \exp[-bBu(2\bar{L}-\bar{x})]\}. \qquad (17)$$

These are inhomogeneous wave equations, with the inhomogeneity arising from the driving radiation from the black wall. The upper signs pertain to waves propagating to the right; the lower signs to waves propagating to the left.

It can be verified that a particular solution of (17), denoted by subscript p, is given by

$$\bar{u}'_p \pm \frac{2\bar{c}'_p}{\gamma-1} = 8ab\frac{Bu^2}{Bo}\frac{|\bar{T}'_w|}{1+b^2Bu^2}\{\exp(-bBu\bar{x})(\pm bBu\cos\bar{t}-\sin\bar{t})$$
$$-\exp[-bBu(2\bar{L}-\bar{x})](\mp bBu\cos\bar{t}-\sin\bar{t})+\sin(\bar{t}\mp\bar{x})[1-\exp(-2bBu\bar{L})]\}. \quad (18)$$

The result for $\bar{u}'_p$, obtained by adding these two expressions, is

$$\bar{u}'_p = -8ab\frac{Bu^2}{Bo}\frac{|\bar{T}'_w|}{1+b^2Bu^2}\{\exp(-bBu\bar{x})-\exp[-bBu(2\bar{L}-\bar{x})]$$
$$-[1-\exp(-2bBu\bar{L})]\cos\bar{x}\}\sin\bar{t}. \quad (19)$$

Note that the particular solution has been chosen such that $\bar{u}'_p$ vanishes at $\bar{x} = 0$. The complementary function, denoted by subscript c, is given by the general solution of the homogeneous wave equation as follows:

$$\bar{u}'_c \pm \frac{2\bar{c}'_c}{\gamma-1} = 2f_\pm(\bar{t}\mp\bar{x}). \quad (20)$$

Here, f_+ and f_- are undetermined functions representing right- and left-running waves, respectively. If we set

$$f_+(z) = -f_-(z) = f(z) \quad (21)$$

so that $\bar{u}'_c$ will vanish at $\bar{x} = 0$, we obtain by addition and subtraction of equation (20)

$$\bar{u}'_c = f(\bar{t}-\bar{x})-f(\bar{t}+\bar{x}), \quad (22)$$

$$\bar{c}'_c = \tfrac{1}{2}(\gamma-1)[f(\bar{t}-\bar{x})+f(\bar{t}+\bar{x})]. \quad (23)$$

Since both $\bar{u}'_c$ and $\bar{u}'_p$ vanish at $\bar{x} = 0$, the velocity $\bar{u}' = \bar{u}'_c+\bar{u}'_p$ satisfies the boundary condition $\bar{u}'(0,t) = 0$. The function f is found by applying the boundary condition $\bar{u}'(\bar{L},\bar{t}) = 0$, which requires that $\bar{u}'_c(\bar{L},\bar{t}) = -\bar{u}'_p(\bar{L},\bar{t})$. To accomplish this we take

$$f(z) = -4ab\frac{Bu^2}{Bo}\frac{|\bar{T}'_w|}{1+b^2Bu^2}[1-\exp(-2bBu\bar{L})]\frac{\cos z}{\tan\bar{L}}. \quad (24)$$

Substituting this expression into (22) and adding the result to expression (19), we obtain finally

$$\bar{u}' = -8ab\frac{Bu^2}{Bo}\frac{|\bar{T}'_w|}{1+b^2Bu^2}\{\exp(-bBu\bar{x})-\exp[-bBu(2\bar{L}-\bar{x})]$$
$$-[1-\exp(-2bBu\bar{L})][\cos\bar{x}-(\sin\bar{x}/\tan\bar{L})]\}\sin\bar{t}. \quad (25)$$

This result for the velocity is acceptably accurate away from resonance. Near resonance, however, where $\bar{L}\approx N\pi$ and hence $\tan\bar{L}\approx 0$, the amplitude of $\bar{u}'$ becomes so large that the small perturbation assumption of the linearized theory is violated. As was discussed in the introduction, this occurs because precisely at resonance the complementary function $\bar{u}'_c$ is unable to satisfy the condition $\bar{u}'(\bar{L},\bar{t}) = 0$ by cancelling the value of $\bar{u}'_p$.

In the laboratory device the radiative driving of the gas is weak. One therefore expects, contrary to the predictions of the simplified linear theory, that the amplitude of the wave motion will actually remain small even at resonance. The most significant portion of the governing equations (16) should then still be the terms of the simplified linear theory. Near resonance, however, one must also take into account both the nonlinear terms and terms due to perturbations in spontaneous emission. These facts together suggest an iterative approach. Any attempt at iteration, however, will clearly fail if one uses as a first approximation the solution of the simplified linear theory given by (25) and the corresponding expression for $\bar{c}'$. Chester suggests using instead the complementary function of the simplified linear theory as given by (22) and (23), that is, with f still undetermined. The form of f will then be found by applying the boundary condition *after* the first iterated solution of the governing equations. The resulting final solution will constitute a uniformly valid first approximation, since the nonlinear terms and the perturbations in spontaneous emission will have been taken into account before the form of f is found. It is sufficient to use only the complementary function for iteration, since, although it does finally remain small at resonance, it still dominates over the particular solution. This is consistent with the near-resonance approximation $\overline{U}' \gg \overline{S}'$ used in § 2, since the particular solution is of order $\overline{S}'$ while the complementary function is of order $\overline{U}'$.

Proceeding on this basis, we first use expressions (22) for $\bar{u}_c'$ and (23) for $\bar{c}_c'$ to evaluate all but the driving term on the right-hand sides of equations (16). The resulting equations are

$$\left(\frac{\partial}{\partial \bar{t}} \pm \frac{\partial}{\partial \bar{x}}\right)\left(\bar{u}' \pm \frac{2\bar{c}'}{\gamma-1}\right) = \pm f'(\bar{t} \mp \bar{x})\,[(\gamma+1)f(\bar{t}\mp\bar{x}) - (3-\gamma)f(\bar{t}\pm\bar{x})]$$

$$- 8ab\frac{Bu^2}{Bo}\left\{|\overline{T}_w'|\cos\bar{t}\{\exp(-bBu\bar{x}) - \exp[-bBu(2\overline{L}-\bar{x})]\}\right.$$

$$-(\gamma-1)\,bBu\int_0^{\overline{L}}[\hat{f}(\bar{t}-\tilde{x}) + \hat{f}(\bar{t}-\tilde{x})]\{\exp(-bBu|\bar{x}-\tilde{x}|)\,\mathrm{sgn}\,(\bar{x}-\tilde{x})$$

$$\left. -\exp[-bBu(2\overline{L}-\bar{x}-\tilde{x})]\}\,d\tilde{x} + [2(\gamma-1)/abBu]\,[f(\bar{t}-\bar{x}) - f(\bar{t}+\bar{x})]\right\}, \quad (26)$$

where $\hat{f}(\bar{t}\pm\bar{x})$ denotes

$$\int_{-\infty}^{\bar{t}} f(\tau\pm\bar{x})\,d\tau.$$

These are inhomogeneous wave equations that must be solved for $\bar{u}'$ in terms of the still undetermined function f. Since they are linear in the unknown $\bar{u}'$ and $\bar{c}'$, the inhomogeneity can be considered as the sum of separate effects and the corresponding contributions of the solution calculated separately.

We begin by calculating the contribution of the nonlinear effects. If subscript n denotes this part of the solution, the pertinent equations are

$$\left(\frac{\partial}{\partial \bar{t}} \pm \frac{\partial}{\partial \bar{x}}\right)\left(\bar{u}_n' \pm \frac{2\bar{c}_n'}{\gamma-1}\right) = \pm f'(\bar{t}\mp\bar{x})\,[(\gamma+1)f(\bar{t}\mp\bar{x}) - (3-\gamma)f(\bar{t}\pm\bar{x})]. \quad (27)$$

These are readily integrated through introduction of the characteristic coordinates $\alpha = \bar{t}-\bar{x}$ and $\beta = \bar{t}+\bar{x}$. We thus obtain

$$\bar{u}_n' = \tfrac{1}{2}(\gamma+1)\,\bar{x}[f(\bar{t}-\bar{x})f'(\bar{t}-\bar{x})+f(\bar{t}+\bar{x})f'(\bar{t}+\bar{x})] - \tfrac{1}{4}(3-\gamma)\,[f'(\bar{t}-\bar{x})\hat{f}(\bar{t}+\bar{x})-f'(\bar{t}+\bar{x})\hat{f}(\bar{t}-\bar{x})]. \quad (28)$$

Note that $\bar{u}_n'$ vanishes at $\bar{x} = 0$.

The contribution from perturbations in spontaneous emission, denoted by subscript r, is calculated from (26) with only the spontaneous-emission terms retained on the right, that is,

$$\left(\frac{\partial}{\partial \bar{t}} \pm \frac{\partial}{\partial \bar{x}}\right)\left(\bar{u}_r' \pm \frac{2\bar{c}_r'}{\gamma-1}\right) = +8ab\frac{Bu^2}{Bo}\left\{(\gamma-1)\,bBu\int_0^{\bar{L}}[f(\bar{t}-\tilde{\bar{x}})+f(\bar{t}+\tilde{\bar{x}})] \times \{\exp(-bBu|\bar{x}-\tilde{\bar{x}}|)\operatorname{sgn}(\bar{x}-\tilde{\bar{x}})-\exp[-bBu(2\bar{L}-\bar{x}-\tilde{\bar{x}})]\}\,d\tilde{\bar{x}} - [2(\gamma-1)/abBu]\,[f(\bar{t}-\bar{x})-f(\bar{t}+\bar{x})]\right\}. \quad (29)$$

To solve these equations a trial solution is assumed of the form

$$\bar{u}_r' \pm \frac{2\bar{c}_r'}{\gamma-1} = \int_0^{\bar{x}}[G_\pm(\bar{t}-\tilde{\bar{x}})+G_\pm(\bar{t}+\tilde{\bar{x}})]\exp[-bBu(x-\tilde{\bar{x}})]\,d\tilde{\bar{x}} - \int_{\bar{x}}^{\bar{L}}[H_\pm(\bar{t}-\tilde{\bar{x}})+H_\pm(\bar{t}+\tilde{\bar{x}})]\exp[-bBu(\tilde{\bar{x}}-\bar{x})]\,d\tilde{\bar{x}} - \int_0^{\bar{L}}[I_\pm(\bar{t}-\tilde{\bar{x}})+I_\pm(\bar{t}+\tilde{\bar{x}})]\exp[-bBu(2\bar{L}-\bar{x}-\tilde{\bar{x}})]\,d\tilde{\bar{x}}. \quad (30)$$

This form is chosen because it has the same structure as the integral terms in (26). The trial solution is then substituted into (29). In the resulting equations integral terms with the same limits are grouped together and combined under the same integral sign. The integrands are then set equal to zero, which results in ordinary differential equations for the functions $G_\pm$, $H_\pm$ and $I_\pm$, typical ones being

$$G_\pm'(z) \mp bBuG_\pm(z) = 8ab^2(\gamma-1)\,(Bu^2/Bo)\hat{f}(z). \quad (31)$$

These equations, as well as those for $H_\pm$ and $I_\pm$, are solved with the use of integrating factors [$\exp(\mp bBuz)$ in the case of (31)]. Additional terms must then be added to the trial solution (30) to account for the non-integral terms on the right-hand sides of equations (29) and for certain non-integral terms that were also generated when the trial solution was substituted into (29). Undetermined complementary functions $h_\pm(\bar{t}\mp\bar{x})$ are also added to the trial solution so that $\bar{u}_r'$ can be made to vanish at $\bar{x} = 0$. Once the expressions for $\bar{u}_r' \pm 2\bar{c}_r'/(\gamma-1)$ have been completed, we add them to obtain $\bar{u}_r'$ and then determine $h_\pm(\bar{t}\mp\bar{x})$ from the foregoing condition. The final expression for $\bar{u}_r'$ is

$$\bar{u}_r' = -4ab^2(\gamma-1)\frac{Bu^3}{Bo}\left\{\int_0^{\bar{x}}\int^{\tilde{\bar{x}}}[\hat{f}(\bar{t}-\eta)-\hat{f}(\bar{t}+\eta)]\{\exp[-bBu(\bar{x}-\eta)] + \exp[-bBu(\bar{x}+\eta-2\tilde{\bar{x}})]\}\,d\eta\,d\tilde{\bar{x}} - \int_{\bar{x}}^{\bar{L}}\int^{\tilde{\bar{x}}}[\hat{f}(\bar{t}-\eta)-\hat{f}(\bar{t}+\eta)]\{\exp[-bBu(\eta-\bar{x})]+\exp[-bBu(2\tilde{\bar{x}}-\bar{x}-\eta)]\}\,d\eta\,d\tilde{\bar{x}} - \int_0^{\bar{L}}\int^{\tilde{\bar{x}}}[\hat{f}(\bar{t}-\eta)-\hat{f}(\bar{t}+\eta)]\{\exp[-bBu(2\bar{L}-\bar{x}-2\tilde{\bar{x}}+\eta)]\}\exp[-bBu(2\bar{L}-\bar{x}-\eta)]\,d\eta\,d\tilde{\bar{x}}\right.$$

$$+\tfrac{1}{2}\int_0^{\bar{L}}\int^{\tilde{\bar{x}}}[\hat{f}(\bar{t}-\bar{x}-\eta)-\hat{f}(\bar{t}-\bar{x}+\eta)+\hat{f}(\bar{t}+\bar{x}-\eta)-\hat{f}(\bar{t}+\bar{x}+\eta)]$$
$$\times\{\exp(-bBu\eta)+\exp[-bBu(2\tilde{\bar{x}}-\eta)]+\exp[-bBu(2\bar{L}-2\tilde{\bar{x}}+\eta)]$$
$$+\exp[-bBu(2\bar{L}-\eta)]\}\,d\eta\,d\tilde{\bar{x}}-2\bar{x}\int^{\bar{x}}[\hat{f}(\bar{t}-\eta)-\hat{f}(\bar{t}+\eta)]\cosh[-bBu(\bar{x}-\eta)]\,d\eta$$
$$+(2\bar{x}/ab^2Bu^2)\,[f(\bar{t}-\bar{x})+f(\bar{t}+\bar{x})]\Big\}. \tag{32}$$

Here we have introduced an integral notation that can be represented generally by

$$\int^{\xi} g(\tau,\eta)\,d\eta.$$

The indefinite integral of $g(\tau,\xi)$ with respect to ξ can be written as

$$\int g(\tau,\xi)\,d\xi = \int^{\xi} g(\tau,\eta)\,d\eta+\alpha(\tau)+\beta,$$

where $\alpha(\tau)$ is an arbitrary function of τ and β is an arbitrary constant. Thus

$$\int^{\xi} g(\tau,\eta)\,d\eta$$

denotes the indefinite integral of $g(\tau,\xi)$ with respect to ξ with both $\alpha(\tau)$ and β set equal to zero.

The contribution from the driving radiation is calculated from (26) with only the terms due to that radiation retained on the right. Since these equations are identical to the governing equations (17) of the simplified linear theory, the solution for this part of the velocity is simply the particular solution $\bar{u}'_p$ of (17), given already by (19).

The solution of (26) for the complete velocity $\bar{u}'$ can now be constructed. It is the sum of the contributions $\bar{u}'_n$ [equation (28)], $\bar{u}'_r$ [equation (32)] and $\bar{u}'_p$ [equation (19)], which together constitute a particular solution of (26), plus the complementary function $\bar{u}'_c = f(\bar{t}-\bar{x})-f(\bar{t}+\bar{x})$. Since each of these parts vanishes at $\bar{x} = 0$, the boundary condition $\bar{u}'(0,t) = 0$ is satisfied.

The condition $u'(\bar{L},\bar{t}) = 0$ remains to be taken care of. Imposition of this condition leads to the following equation governing the function f:

$$f(t-\bar{L})-f(\bar{t}+\bar{L})+\tfrac{1}{2}(\gamma-1)\,\bar{L}[f(\bar{t}-\bar{L})f'(\bar{t}-\bar{L})+f(\bar{t}+\bar{L})f'(\bar{t}+\bar{L})]$$
$$-\tfrac{1}{4}(3-\gamma)\,[f'(\bar{t}-\bar{L})f(\bar{t}+\bar{L})-\hat{f}(\bar{t}+\bar{L})\hat{f}(\bar{t}-\bar{L})]$$
$$+8ab\frac{Bu^2}{Bo}\frac{|\bar{T}'_w|}{1+b^2Bu^2}[1-\exp(-2bBu\bar{L})]\cos\bar{L}\sin\bar{t}-4ab^2(\gamma-1)\frac{Bu^3}{Bo}$$
$$\times\Big\{\tfrac{1}{2}\int_0^{\bar{L}}\int^{\tilde{\bar{x}}}[\hat{f}(\bar{t}-\bar{L}-\eta)-\hat{f}(\bar{t}-\bar{L}+\eta)+\hat{f}(\bar{t}+\bar{L}-\eta)-\hat{f}(\bar{t}+\bar{L}+\eta)]$$
$$\times\{\exp(-bBu\eta)+\exp[-bBu(2\bar{L}-\eta)]+\exp[-bBu(2\tilde{\bar{x}}-\eta)]$$
$$+\exp[-bBu(2\bar{L}-2\tilde{\bar{x}}+\eta)]\}\,d\eta\,d\tilde{\bar{x}}$$
$$-2L\int^{\bar{L}}[\hat{f}(\bar{t}-\eta)-\hat{f}(\bar{t}+\eta)]\cosh[-bBu(\bar{L}-\eta)]\,d\eta$$
$$+\frac{2\bar{L}}{ab^2Bu^2}[f(t-L)+f(t+L)]\Big\} = 0. \tag{33}$$

Following Chester's procedure, we can simplify this equation by taking advantage of the near-resonance approximation $|L-N\pi| \ll 1$. Since f is assumed to be periodic with the period 2π, it follows that

$$f(\bar{t}+N\pi) = f(\bar{t}-N\pi). \tag{34}$$

With the definition $\Delta = \bar{L}-N\pi$, this can be written as

$$f(\bar{t}+\bar{L}-\Delta) = f(\bar{t}-\bar{L}+\Delta), \tag{35}$$

and the near-resonance condition is $|\Delta| \ll 1$. Thus (35) can be expanded to first order in Δ to obtain

$$f(\bar{t}+\bar{L}) \approx f(\bar{t}-\bar{L})+\Delta f'(\bar{t}-\bar{L})+\Delta f'(\bar{t}+\bar{L}). \tag{36}$$

Differentiation of this equation shows that $f'(\bar{t}+\bar{L}) = f'(\bar{t}-\bar{L})+O(\Delta)$, and this result can be used to rewrite (36) to first order as

$$f(\bar{t}+\bar{L}) \approx f(\bar{t}-\bar{L})+2\Delta f'(\bar{t}-\bar{L}). \tag{37}$$

Since $|\Delta|$ is small, we have to a first approximation

$$\Delta \approx \tan(\bar{L}-N\pi) = \tan\bar{L}, \tag{38}$$

which can be used to write (37) as

$$f(\bar{t}+\bar{L}) \approx f(\bar{t}-\bar{L})+2\tan\bar{L}f'(\bar{t}-\bar{L}). \tag{39}$$

This near-resonance approximation is used in (33) to simplify in turn the terms derived from $\bar{u}'_c$, $\bar{u}'_n$ and $\bar{u}'_r$. In each case only the *leading* term in $\Delta \approx \tan\bar{L}$ is retained. One finds that the leading terms from $\bar{u}'_n$ and $\bar{u}'_r$ are independent of Δ, while that from $\bar{u}'_c$ contains the first power of Δ.

In the resulting equation a further simplification is achieved by using throughout the transformation $\bar{t}_1 = \bar{t}-\bar{L}$. The factor $\sin\bar{t}$ in the term derived from $\bar{u}'_p$ can then be written, with the use of the relation $\bar{L} = N\pi+\Delta$ and to the lowest order in Δ, as

$$\sin\bar{t} = \sin(\bar{t}_1+\bar{L}) \approx \sin\bar{t}_1/\cos\bar{L}. \tag{40}$$

The motivation for this, as we shall see presently, is to recover the solution of the simplified linear theory away from resonance. The subscript on $\bar{t}_1$ is hereafter dropped.

The relation $\bar{L} = N\pi+\Delta$ is also used to replace $\bar{L}$ by $N\pi$ when it appears in the integral terms. This is equivalent to substituting $N\pi+\Delta$ for $\bar{L}$, expanding for small Δ and retaining only the leading term.

Once the above simplifications to (33) have been carried out, the resulting equation is integrated with respect to $\bar{t}$. This results in

$$\begin{aligned} c+\tfrac{1}{2}\cos\bar{t} = [F(\bar{t})-2r/\pi]^2-2\delta\Big\{Bu^2\int_0^{N\pi}\int^{\tilde{\bar{x}}}\int[\hat{F}(t-\eta)-\hat{F}(\bar{t}+\eta)] \\ \times\{\exp(-bBu\eta)+\exp[-bBu(2N\pi-\eta)]+\exp[-bBu(\eta-2\tilde{\bar{x}})] \\ +\exp[-bBu(2N\pi-2\tilde{\bar{x}}+\eta)]\}\,dt\,d\eta\,d\tilde{\bar{x}}-2N\pi Bu^2\int^{N\pi}\int[\hat{F}(\bar{t}+N\pi-\eta) \\ -\hat{F}(\bar{t}+N\pi+\eta)]\cosh[-bBu(N\pi-\eta)]\,d\bar{t}\,d\eta+\frac{4N\pi}{ab^2}\int F(\bar{t})\,dt\Big\}, \end{aligned} \tag{41}$$

where we have introduced the following definitions:

$$r \equiv \frac{\pi \tan \bar{L}}{(\gamma+1)\bar{L}\epsilon^{\frac{1}{2}}}, \quad \delta \equiv \frac{\gamma-1}{\gamma+1}\frac{4ab^2Bu}{\bar{L}\epsilon^{\frac{1}{2}}Bo}, \quad F(\bar{t}) \equiv \frac{f(\bar{t})}{\epsilon^{\frac{1}{2}}}, \tag{42)-(44}$$

where
$$\epsilon \equiv \frac{32ab}{(\gamma+1)\bar{L}}\frac{Bu^2}{Bo}\frac{|\bar{T}_w'|}{1+b^2Bu^2}[1-\exp(-2bBu\bar{L})]. \tag{45}$$

The integration constant c in (41) must be chosen so that the mean value of F over an interval of 2π is zero. This condition is a consequence of the fact that over one cycle the integral of $\bar{u}'$ with respect to time must be zero so that gas particles return periodically to the same position.

The integrations indicated in (41) can be carried out formally by representing the function F by a Fourier series as follows:

$$\begin{aligned} F(\bar{t}) &= \sum_{n=1}^{\infty}(a_n \cos n\bar{t} + b_n \sin n\bar{t}) \\ &= \frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^{2\pi} F(\xi)\cos[n(\bar{t}-\xi)]\,d\xi. \end{aligned} \tag{46}$$

The constant term in the series is omitted because the mean value of F is zero. Expression (46) is used to rewrite (41) as

$$c+\tfrac{1}{2}\cos\bar{t} = \left[F(\bar{t})-\frac{2r}{\pi}\right]^2 - 2\delta\int_0^{2\pi} F(\xi)\,K(\bar{t}-\xi; Bu, N)\,d\xi, \tag{47}$$

where

$$\begin{aligned} K(z; Bu, N) = \sum_{n=1}^{\infty} \sin nz \Bigg\{ & -\frac{2b^2Bu^2}{n^2\pi}\int_0^{N\pi}\int^{\tilde{\bar{x}}} \sin nz \\ & \times\{\exp(-bBu\eta) + \exp[-bBu(2N\pi-\eta)] + \exp[-bBu(\eta-2\tilde{\bar{x}})] \\ & + \exp[-bBu(2N\pi-2\tilde{\bar{x}}+\eta)]\}\,d\eta\,d\tilde{\bar{x}} + \frac{4Nb^2Bu^2(-1)^{Nn}}{n^2} \\ & \times\int^{N\pi} \sin n\eta \cosh[-bBu(N\pi-\eta)]\,d\eta + \frac{4N}{nab^2}\Bigg\}. \end{aligned} \tag{48}$$

With the integrations indicated in this expression carried out, we obtain

$$\begin{aligned} K(z; Bu, N) = \sum_{n=1}^{\infty}\frac{4\sin nz}{n^2+b^2Bu^2}\Bigg\{ & \frac{bBu^3}{n\pi(n^2+b^2Bu^2)}[1-\exp(-2bBuN\pi)] \\ & + \frac{N^2[n^2+(1-a)b^2Bu^2]}{ab^2n}\Bigg\}. \end{aligned} \tag{49}$$

For large values of n the terms in this series behave like

$$(4N/ab^2)(1/n - ab^2Bu^2/n^3)\sin nz.$$

The convergence is improved considerably by subtracting this quantity from each term in the series, and compensating by adding the analytical sum

$$(4N/ab^2)\sum_{n=1}^{\infty}(1/n - ab^2Bu^2/n^3)\sin nz$$

to the expression for $K(z)$. In this way we obtain

$$\begin{aligned} K(z; Bu, N) = 4bBu^3\sum_{n=1}^{\infty}\frac{\sin nz}{n(n^2+b^2Bu^2)}\left\{\frac{[1-\exp(-2bBuN\pi)]}{\pi(n^2+b^2Bu^2)}+\frac{NbBu}{an^2}\right\} \\ + \frac{NBu^2}{3}(\pi-z)^3 + \left(\frac{2N}{ab^2}-\frac{N\pi^2Bu^2}{3}\right)(\pi-z). \end{aligned} \tag{50}$$

Note that the terms of the series in this expression behave like $n^{-5}\sin nz$.

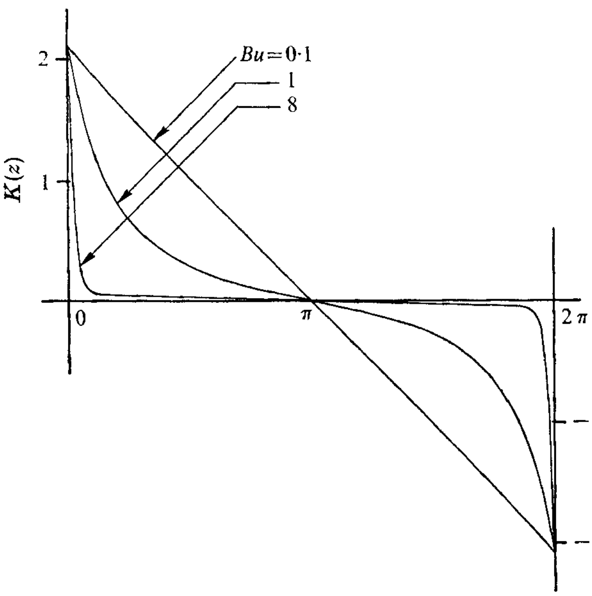

FIGURE 2. The kernel $K(z)$ calculated from equation (50) for the first resonant frequency ($N = 1$) and with $a = 1$, $b = \sqrt{3}$.

The function $K(z)$, calculated from (50), is shown in figure 2 for the first resonant condition $N = 1$, and for $Bu = 0{\cdot}1$, 1 and 8. Since $K(z)$ has period 2π, it is displayed only in the interval 0 to 2π. We see that $K(z)$ depends strongly on the value of Bu. Note also that it is an odd function around the point $z = \pi$, and that it has discontinuous jumps at $z = 0$ and 2π.

The nonlinear integral equation (47) is the required equation that governs the response of the gas. The first term is the integration constant that remains to be chosen so that the mean value of F is zero. The second term arises from the driving radiation from the black wall. The third contains both the nonlinear and linear terms of the gasdynamics. The integral term is from the perturbation in spontaneous emission.

We now wish to ascertain the criteria that must be satisfied for the governing equation (47) to be valid. This will also make clear the role of the various small parameters of the problem. Basic assumptions for our application of Chester's method are that the nonlinear contribution $\bar{u}'_n$, the radiative contribution $\bar{u}'_r$ and the particular solution $\bar{u}'_p$ are small compared with the complementary function $\bar{u}'_c$ (except, of course, in the neighbourhood of a node of $\bar{u}'_c$). A close examination of these terms (for details, see Eninger 1971) shows that we have $\bar{u}'_n \ll \bar{u}'_c$, $\bar{u}'_r \ll \bar{u}'_c$ and $\bar{u}'_p \ll \bar{u}'_c$ only if, respectively, $\epsilon^{\frac{1}{2}} \ll 1$, $1/Bo \ll 1$ and $\tan \bar{L} \ll 1$. In order to specify the relative size of these three small parameters, the two similarity parameters δ and r are required (δ [equation (43)] is proportional to the ratio of $1/Bo$ to $\epsilon^{\frac{1}{2}}$; r [equation (42)] is proportional to the ratio of $\tan \bar{L}$ to $\epsilon^{\frac{1}{2}}$). In physical terms, ϵ [equation (45)] is a measure of the strength of the driving radiation, δ is a measure of the importance of perturbations in spontaneous emission, and r is a normalized deviation from resonance.

Because the near-resonance assumption $\tan \bar{L} \ll 1$ was used in the derivation of (47) (not only to ensure that $\bar{u}'_p \ll \bar{u}'_c$, but also so that approximation (39) could be used), one would expect this equation to be valid only near resonance. This would have been the case, except that the arbitrary substitution of (40), valid within the near-resonance approximation, was introduced with the specific purpose of modifying (47) so that it is still valid away from resonance as well. To see this, consider that away from resonance $\tan \bar{L}$ is no longer small, so that we have $r \to \infty$. In this limit the nonlinear and integral terms can be dropped from (47). The solution of the resulting equation is identical to expression (24), which is the solution of the simplified linear theory valid away from resonance.

4. Solution of the nonlinear integral equation

Before proceeding with the general solution of (47), it is instructive to consider the limiting cases $\delta \to \infty$ and $\delta \to 0$.

Consideration of the order of magnitude of the terms of (47) shows that in the limit $\delta \to \infty$ the nonlinear term is negligible compared with the integral term. Thus the valid governing equation in this limit is a linear integral equation with a harmonic forcing term. One expects, then, that the solution is also harmonic. By assuming a sinusoidal form for $F(\bar{t})$, we are lead to the solution

$$F(\bar{t}) = \tfrac{1}{2}[(2\delta K_1)^2 + (4r/\pi)^2]^{-\frac{1}{2}} \sin(t-\phi), \tag{51}$$

where

$$\phi = \tan^{-1}(2r/\delta K_1 \pi^2), \tag{52}$$

and K_1 is the coefficient of $\sin z$ in the first term of the series in (49). Since this solution is valid in the limit where nonlinear terms are negligible, it should be comparablewith the Long–Vincenti (1967) linear theory. Unlike the present theory, however, the Long–Vincenti theory is not restricted to the condition $1/Bo \ll 1$. Our result does, in fact, agree exactly with an analytical result that Compton (1969) obtained for the Long–Vincenti theory in the limit $1/Bo \to 0$.

In the limit $\delta \to 0$ the integral term in (47) – and hence the effect of spontaneous emission – vanishes. The resulting equation is

$$b^2 + \cos^2 \tfrac{1}{2}\bar{t} = [F(\bar{t}) - 2r/\pi]^2, \tag{53}$$

where the identity $\cos \bar{t} = 2\cos^2 \frac{1}{2}\bar{t} - 1$ has been introduced and the undetermined constant b^2 is related to c by

$$b^2 = c - \tfrac{1}{2}. \tag{54}$$

Equation (53) is identical to the equation that Chester (1964) obtains as governing piston-driven resonant wave motion of an inviscid gas. Thus we see that in the limit $\delta \to 0$ the gas responds in the same manner whether it is driven by a piston or by a sinusoidally varying radiative flux.†

† This final statement applies for arbitrary δ as well. For if, in the radiative problem, we choose to drive the gas with a small amplitude piston motion, the governing equation for arbitrary δ turns out to be formally identical to (47), which governs the radiatively driven case.

The solution of (53), which follows Chester, is reproduced here, because the concepts involved will be needed in the general solution of (47). We begin by formally solving (53) for $F(\bar{t})$, which results in

$$F(\bar{t}) = 2r/\pi \pm [b^2 + \cos^2 \tfrac{1}{2}\bar{t}]^{\frac{1}{2}}. \tag{55}$$

The sign of the square root and the value of b^2 must then be taken so that the mean value of F is zero. To this end expression (55) is integrated from 0 to 2π and set equal to zero, which leads to

$$\int_0^{2\pi} F(\bar{t})\, d\bar{t} = 4r \pm \int_0^{2\pi} [b^2 + \cos^2 \tfrac{1}{2}\bar{t}]^{\frac{1}{2}}\, d\bar{t} = 0. \tag{56}$$

This equation is rewritten as

$$\begin{aligned} |r| &= (1+b^2)^{\frac{1}{2}} \int_0^{\frac{1}{2}\pi} \left[1 - \frac{1}{1+b^2}\sin^2\tau\right]^{\frac{1}{2}} d\tau \\ &= (1+b^2)^{\frac{1}{2}} E[(1+b^2)^{-\frac{1}{2}}, \tfrac{1}{2}\pi], \end{aligned} \tag{57}$$

where E is the complete elliptic integral of the second kind. Equation (57) is a transcendental equation that gives the undetermined constant b^2 in terms of r. Since the right-hand side of (53) is always positive, the left-hand side must be positive as well. As a consequence, we have the condition $b^2 \geqslant 0$. All values of $b^2 \geqslant 0$, however, correspond in (57) to values of $|r| \geqslant 1$. Thus only for $|r| \geqslant 1$ is it possible to find a continuous solution with a zero mean value. In this case, the sign of the radical in (55) is chosen such that (56) is satisfied. This can be accomplished by writing the solution for $|r| \geqslant 1$ as

$$F(\bar{t}) = 2r/\pi - \operatorname{sgn}(r)\, [b^2 + \cos^2 \tfrac{1}{2}\bar{t}]^{\frac{1}{2}}. \tag{58}$$

For $|r| < 1$, no continuous solution exists that has a zero mean value. We must therefore consider the possible construction of a discontinuous solution from segments of the two solutions given by (55), the segments being connected by discontinuities that represent shock waves. The constant b^2 and the position of the discontinuities must be selected so that the solution has a zero mean value and a period of 2π.

Since b^2 must be greater than or equal to zero, we have two cases to consider. Either b^2 equals zero, in which case the two solutions given by (55) have a point in common; or b^2 is greater than zero, and no common point exists. In the latter case, if we introduce a discontinuous jump from the lower solution, given by the minus sign in (55), to the upper solution, given by the plus sign, we are faced with a dilemma. As will be shown presently, F is proportional to the pressure on the reflecting wall. Thus a jump from the lower solution to the upper represents a compressive shock. Since F is periodic, we must return to the lower solution, but the only way to accomplish this is with another discontinuous jump. Such a jump from the upper solution to the lower represents a rarefaction shock, which is a physical impossibility. Thus we have no choice but to take $b^2 = 0$. With $b^2 = 0$ equation (55) becomes

$$F(\bar{t}) = 2r/\pi \pm |\cos \tfrac{1}{2}\bar{t}|. \tag{59}$$

The upper and lower solutions now have a point in common whenever $\bar{t}$ is an integer multiple of π. If we begin on the lower solution and jump to the upper solution by means of a discontinuity representing a compressive shock, we can now return to the lower solution through the common point, that is, by means of a continuous expansion. If consecutive shock waves occur at $\bar{t} = \theta$ and $\bar{t} = 2\pi + \theta$, then F can be written for the present case $|r| < 1$ as

$$F(\bar{t}) = 2r/\pi + \cos \tfrac{1}{2}\bar{t}, \tag{60}$$

for

$$\theta \leqslant \bar{t} \leqslant 2\pi + \theta. \tag{61}$$

Outside this interval F is defined as the periodic extension of expression (60). The position θ of the shock wave is determined by the condition that F must have zero mean value over the interval (61). This yields the result

$$\theta = 2\sin^{-1} r. \tag{62}$$

To be specific, we shall take $-\frac{1}{2}\pi \leqslant \sin^{-1} r \leqslant \frac{1}{2}\pi$.

We can now face the task of solving (47) for arbitrary δ. The arguments to be employed are the same as those used in Chester's solution for the case $\delta = 0$. The steps are more involved, however, because it is not possible to solve the general equation explicitly for F. To begin, since the integrand term has period 2π, equation (47) can be rewritten as

$$\left[F(\bar{t}) - \frac{2r}{\pi}\right]^2 = c + \tfrac{1}{2}\cos \bar{t} + 2\delta \int_{\theta}^{2\pi+\theta} F(\xi)\, K(\bar{t}-\xi)\, d\xi, \tag{63}$$

where θ gives the position of a shock wave if one exists. As with the limiting solution (60), equation (63) is taken to govern F in the interval $\theta \leqslant \bar{t} \leqslant 2\pi + \theta$. One expects shock waves to exist for certain values of r if δ is sufficiently small, since the solution in the limit $\delta \to 0$ was seen to contain such waves. On the other hand, one does not expect shock waves for any r if δ is sufficiently large, since the solution for $\delta \to \infty$ did not contain such waves. In the latter case, (63) still governs F, but θ is considered merely as an arbitrary constant. In either case, the mean value of F must be zero, which gives the condition

$$\int_{\theta}^{2\pi+\theta} F(\bar{t})\, d\bar{t} = 0. \tag{64}$$

If no shock exists, then (63) and (64) are sufficient to define both F and the undetermined constant c.

If a shock wave does exist, additional equations are needed to express the condition that no rarefaction shock is permissible. By taking the square root of equation (63), we obtain for the two solution branches

$$\left[F(\bar{t}) - \frac{2r}{\pi}\right] = \pm \left[c + \tfrac{1}{2}\cos \bar{t} + 2\delta \int_{\theta}^{2\pi+\theta} F(\xi)\, K(\bar{t}-\xi)\, d\xi\right]^{\frac{1}{2}}. \tag{65}$$

From the original equation (63), the expression within the radical must be always greater than or equal to zero. In addition, both branches of the solution must have a point in common in order that a continuous transition from the upper branch to the lower branch is possible. This point will occur when the expression within the radical vanishes. Thus the necessary value of c can be written as

$$c = -\tfrac{1}{2}\cos \bar{t}_m - 2\delta \int_{\theta}^{2\pi+\theta} F(\xi)\, K(\bar{t}_m - \xi)\, d\xi, \tag{66}$$

where $\bar{t}_m$ is the value of $\bar{t}$ at which the expression inside the radical is a minimum. Thus $\bar{t}_m$ must satisfy the condition for a minimum, that is,

$$\left\{\frac{d}{d\bar{t}}\left[c+\tfrac{1}{2}\cos\bar{t}+2\delta\int_{\theta}^{2\pi+\theta}F(\xi)\,K(\bar{t}-\xi)\,d\xi\right]\right\}_{\bar{t}=\bar{t}_m}=0. \tag{67}$$

Carrying out the differentiation, we obtain

$$-\tfrac{1}{2}\sin\bar{t}_m+2\delta\int_{\theta}^{2\pi+\theta}F(\xi)\,K'(\bar{t}_m-\xi)\,d\xi=0. \tag{68}$$

An alternative equation for $\bar{t}_m$ can be obtained by noting in (65) that since the right-hand side vanishes at $\bar{t}=\bar{t}_m$, the left-hand side must vanish there as well. Thus we write

$$F(\bar{t}_m)-2r/\pi=0. \tag{69}$$

Equations (63), (64), (66) and either (68) or (69) constitute four equations that govern F, c, θ and $\bar{t}_m$ for the case when shock waves exist. When shock waves do not exist, θ is chosen arbitrarily, and the last two equations, (66) and (68) or (69), are dropped. In either case, the main equation (63) is a nonlinear integral equation. The strategy that will be employed to solve these equations is to reduce them to a set of purely linear equations by means of the method of parametric differentiation.

This method was first applied to problems in fluid mechanics by Rubbert & Landahl (1967). They demonstrated the method by applying it to the Falkner–Skan boundary-layer equation and to the problem of airfoils at transonic speeds. The first application to problems of radiative gasdynamics was by Jischke & Baron (1969). They used the method to obtain solutions for radiating gas flow in the stagnation region of a blunt body. The central idea of parametric differentiation is to calculate the small change that a solution makes away from a known solution as the result of a small change in a suitable parameter. This provides a new solution corresponding to the increased value of the parameter, and the process is repeated. In this manner, the solution is found for the desired range of the parameter. The advantage of parametric differentiation is that the equations that govern the change of the solution with respect to a change in the parameter are linear.

In the present problem we apply the method by considering the solution as a function of δ, that is, we regard the four unknowns as $F(\bar{t};\delta)$, $c(\delta)$, $\theta(\delta)$ and $\bar{t}_m(\delta)$. We now calculate the change in these quantities as a result of a small change in δ by differentiating (63), (64) and (66). In the equation resulting from (66), the coefficient of $\partial\bar{t}_m/\partial\delta$ is identical to the left-hand side of (68); the term with $\partial\bar{t}_m/\partial\delta$ therefore disappears. With the aid of the fact that $K(z)$ is periodic with period 2π, and with introduction of the notation ΔF for the magnitude of the discontinuity in F [i.e. $F(\theta_+)-F(\theta_-)$], the differentiated equations can be written as

$$\left[F(\bar{t};\delta)-\frac{2r}{\pi}\right]\frac{\partial F}{\partial\delta}(\bar{t};\delta)-\delta\int_{\theta}^{2\pi+\theta}\frac{\partial F}{\partial\delta}(\xi;\delta)\,K(\bar{t}-\xi)\,d\xi-\frac{1}{2}\frac{\partial c}{\partial\delta}$$
$$+\delta K(\bar{t}-\theta)\,\Delta F\frac{\partial\theta}{\partial\delta}=\int_{\theta}^{2\pi+\theta}F(\xi;\delta)\,K(\bar{t}-\xi)\,d\xi, \tag{70}$$

$$\int_{\theta}^{2\pi+\theta} \frac{\partial F}{\partial \delta}(\xi;\delta)\, d\xi - \Delta F \frac{\partial \theta}{\partial \delta} = 0, \tag{71}$$

$$\delta \int_{\theta}^{2\pi+\theta} \frac{\partial F}{\partial \delta}(\xi;\delta)\, K(\bar{t}_m-\xi)\, d\xi + \frac{1}{2}\frac{\partial c}{\partial \delta} - \delta K(\bar{t}_m-\theta)\,\Delta F \frac{\partial \theta}{\partial \delta} = -\int_{\theta}^{2\pi+\theta} F(\xi;\delta)\, K(\bar{t}_m-\xi)\, d\xi. \tag{72}$$

For simplicity, the dependence of θ and $\bar{t}_m$ on δ is not indicated specifically in these equations.

In each cycle of the step-by-step procedure, the results from the previous cycle are first substituted for $F(\bar{t};\delta)$, $\theta(\delta)$ and $\bar{t}_m(\delta)$ in the above equations. These then become three *linear* integral equations for the derivatives $\partial F/\partial\delta$, $\partial c/\partial\delta$ and $\partial\theta/\partial\delta$. Once this set of equations has been solved (by a method to be described presently), a neighbouring solution is calculated from

$$F(t;\delta+\Delta\delta) = (\partial F/\partial\delta)\,\Delta\delta + F(\bar{t};\delta), \tag{73}$$

$$c(\delta+\Delta\delta) = (\partial c/\partial\delta)\,\Delta\delta + c(\delta), \tag{74}$$

$$\theta(\delta+\Delta\delta) = (\partial\theta/\partial\delta)\,\Delta\delta + \theta(\delta). \tag{75}$$

With the function $F(\bar{t};\delta+\Delta\delta)$ obtained from (73), the value of $\bar{t}_m(\delta+\Delta\delta)$ is found from (69).

To start the step-by-step procedure, we use the solution obtained earlier for $\delta = 0$. With the foregoing scheme, solutions are then generated for larger and larger δ. These at first contain shock waves. As we have seen, however, for $\delta \to \infty$ the solutions must be continuous. Thus a point is presumably reached in the step-by-step process at which the shock waves disappear. That is, at a certain value $\delta = \delta_{\text{crit}}$, ΔF vanishes. Since the coefficients of $\partial\theta/\partial\delta$ in (70), (71) and (72) each contain the factor ΔF, this is a singular point for these equations. For $\delta > \delta_{\text{crit}}$, it is necessary to use only (70) and (71) with ΔF set equal to zero and θ considered as an arbitrary constant. In this case we need calculate only F and c.

For $\delta > \delta_{\text{crit}}$ is not possible to continue from the solution previously obtained for $\delta = \delta_{\text{crit}}$. This follows from the fact that δ_{crit} is a singular point of the equations for $\delta > \delta_{\text{crit}}$ as well. One can, however, begin the procedure anew by starting with a known solution for δ far above δ_{crit}. Solutions are then generated for decreasing values of δ by using a negative increment $\Delta\delta$. A starting solution for $\delta \gg \delta_{\text{crit}}$ is found from (63) and (64). Since the nonlinear term is small for large values of δ, these equations are readily solved by iteration, with the previously obtained solution for $\delta \to \infty$ used as a first approximation. In this iterative process the equations for each successive approximation are linear.

On may wonder why an iteration procedure such as this is not used to find the solution for all values of δ. To speed the process the converged solution for one value of δ could be used as the first approximation for a neighbouring value of δ. Such a scheme was in fact tried. While it was successful for both large and small values of δ, there is a range of δ around the critical value within which the iterations do not converge.

It remains to describe the method used to solve the linear integral equations (70)–(72). For this purpose we take n equally spaced points on the interval

$\theta \leqslant \bar{t} \leqslant 2\pi + \theta$, with $\bar{t}$ at the ith point denoted by $\bar{t}_i$. The trapezoidal integration formula is then used to express each integral term as a weighted sum of the value of the integrand at the n points. (Allowance must be made for the discontinuity of the kernel K; for details, see Eninger 1971.) Since the resulting equations hold for any $\bar{t}$, this variable can be set equal in turn to its value at each point in the interval. This procedure reduces the set of linear integral equations to a set of simultaneous linear algebraic equations for the value of the solution at the n points. These equations are solved on a computer.

5. Results and discussion

The solutions that will be presented are restricted to the frequency range $|r| \leqslant 1$ around the first resonant frequency ($N = 1$). Figure 3 gives $F(\bar{t})$ for three values of Bu (0·1, 1, and 8) when the system is precisely at resonance; figure 4 displays $F(\bar{t})$ for the three near-resonance situations $r = 0{\cdot}5$, $0{\cdot}9$ and 1, all for the intermediate value of $Bu = 1$. In each case, the method of parametric differentiation generates solutions for numerous values of δ, but only about every fifth solution is shown. The solutions for the different values of Bu are calculated on the basis of the kernels displayed in figure 2.† The kernels, which are calculated from (50), depend not only on Bu and N, but also on the constants a and b of the exponential approximation. Here a is taken as 1 and b as $\surd 3$. It is only necessary to calculate solutions for positive values of r, because it can be shown from the governing equation (47) that, if $F(\bar{t})$ is a solution for r, then $-F(-\bar{t})$ is a solution for $-r$. This is a consequence of the fact that the kernel $K(z)$ is an odd function about $z = \pi$.

The function F admits more than one physical interpretation. For example, near resonance the velocity is given to a first approximation by the complementary function alone. That is, although the nonlinear and radiative contributions and the particular solution are essential for obtaining the equation that governs F, once F is found these contributions can be ignored. Thus the velocity is given by
$$\bar{u}' = \epsilon^{\frac{1}{2}}[F(\bar{t}-\bar{x}) - F(\bar{t}+\bar{x})].$$
From this we obtain, with the use of the linearized form of the momentum equation (2) and the periodic property of F, the following expression for the perturbation pressure at $\bar{x} = \bar{L}$:
$$\bar{p}'(L,t) = 2\gamma\epsilon^{\frac{1}{2}}F(\bar{t}-\bar{L}).$$
Thus near resonance F can be thought of as describing the time variation of pressure on the reflecting wall.

There is one exception for which the foregoing expression is invalid. For the limit $Bu \to 0$, careful consideration of the order of magnitude of the contribution to the pressure from the particular solution shows that this contribution cannot

† It is apparent from figure 2 that, as Bu becomes large, the kernel develops sharp bends near the end points of the interval. The kernel for $Bu = 8$ was calculated rather than that for $Bu = 10$, since the former can be more accurately represented by its value at a reasonable number of points. In this case, 81 points were used in the interval $\theta \leqslant \bar{t} \leqslant 2\pi$ for the solution of the linear integral equations. For the other two values of Bu, 41 points were used. The solutions were carried out on the IBM 360 computer at Stanford University. The computer time needed to generate a set of solutions was always less than 6 min.

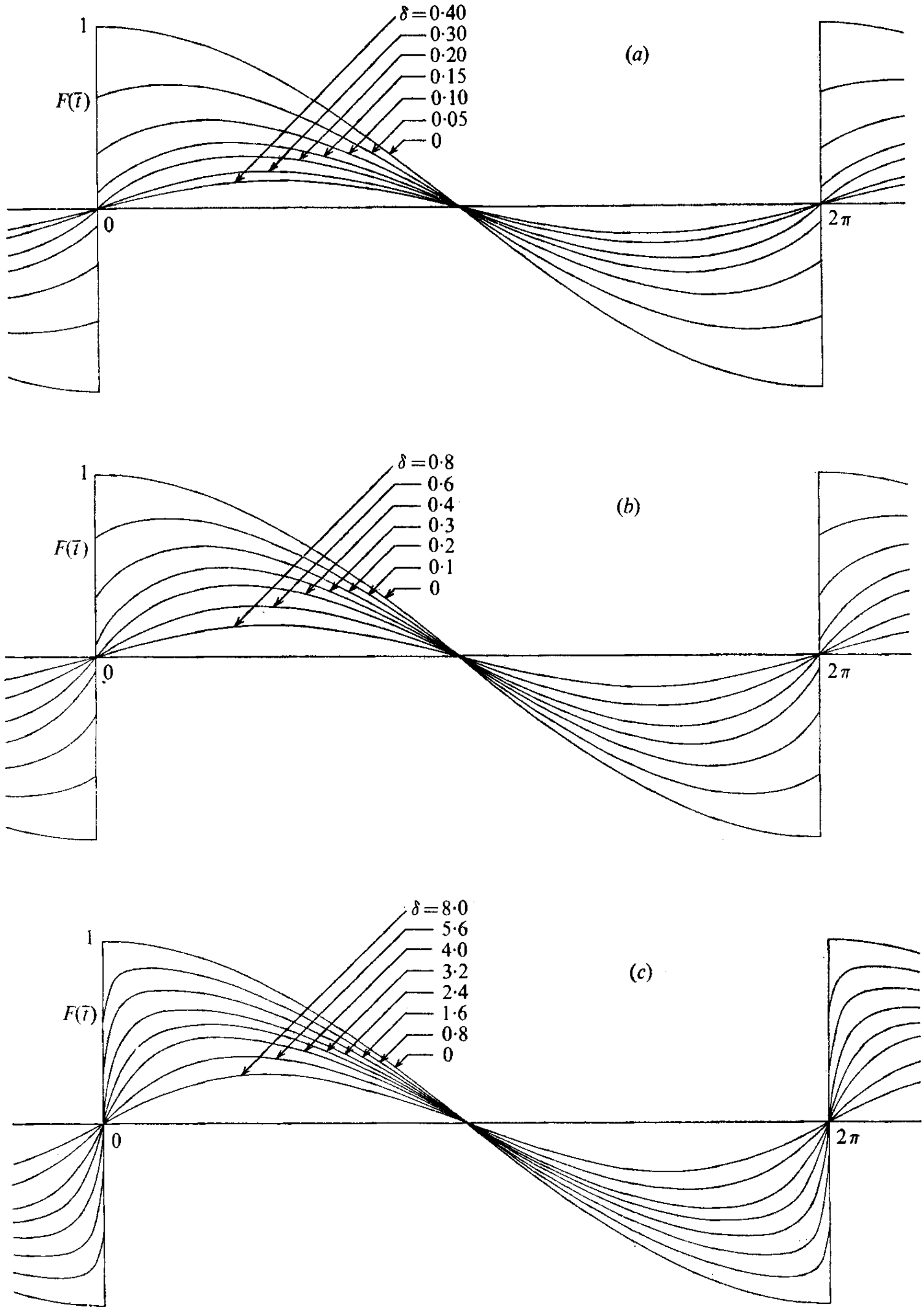

FIGURE 3. Solutions precisely at the first resonance ($r = 0$) for (*a*) $Bu = 0{\cdot}1$, (*b*) $Bu = 1$ and (*c*) $Bu = 8$ ($N = 1$, $a = 1$, $b = \surd 3$).

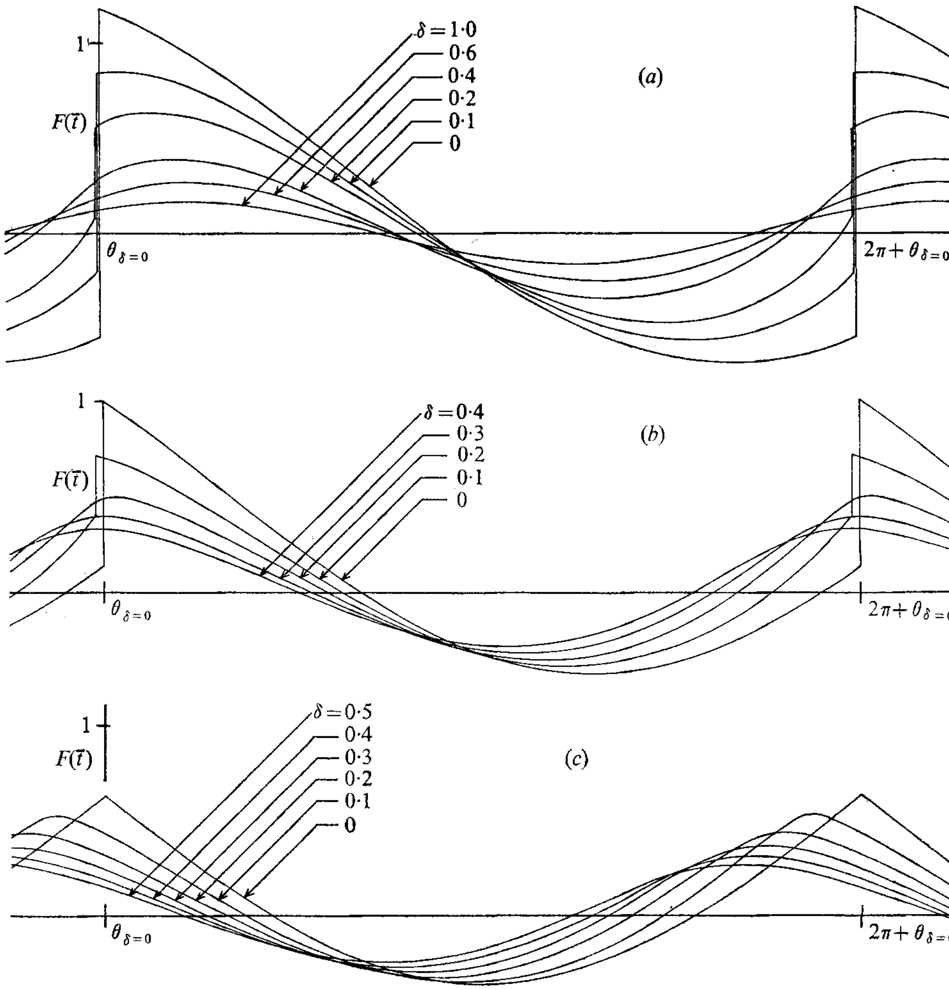

FIGURE 4. Solutions near the first resonant frequency for $Bu = 1$ ($N = 1$, $a = 1$, $b = 3$). (*a*) $r = 0{\cdot}5$, $\theta_{\delta=0} = -\frac{2}{3}\pi$. (*b*) $r = 0{\cdot}9$, $\theta_{\delta=0} = 0{\cdot}917$. (*c*) $r = 1$, $\theta_{\delta=0} = 0$.

be neglected. In this limit the situation is optically thin, and the effect of the driving radiation is to heat the gas almost uniformly along the length of the tube. Uniform heating does not excite wave motion. Hence, even at resonance the contribution to the pressure from the wave motion, that is, from the complementary function, can be smaller than the contribution from the radiative heating, that is, from the particular solution.

Let us first consider the results precisely at resonance (figure 3). The most noticeable result of an increase in the relative level of the perturbations in spontaneous emission (increased δ) is that, for a fixed relative strength at the driving radiation (fixed ϵ), the strength of the shock wave rapidly diminishes. Physically, this is due to an increase in the net heat flux emitted by the heated gas behind the shock and absorbed by the cooler gas ahead of it. In the optically relatively thick case of $Bu = 8$ [figure 3(*c*)], the effect of this radiative transfer

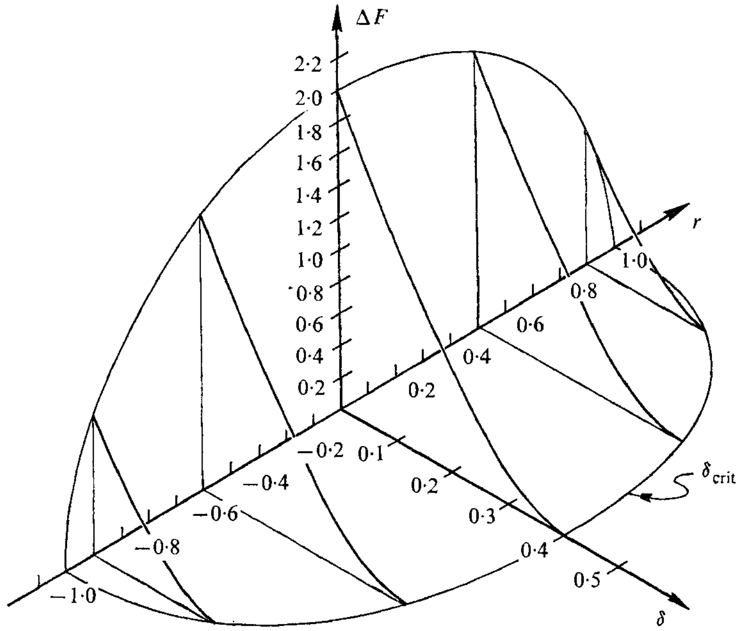

FIGURE 5. Relationship between the magnitude of the discontinuity ΔF, the deviation from resonance r, and the radiative parameter δ ($Bu = 1$, $N = 1$, $a = 1$, $b = \sqrt{3}$).

is confined to the vicinity of the shock; in the thinner case of $Bu = 0{\cdot}1$ [figure 3(*a*)], the effect spreads well away from the shock.

Increase in spontaneous emission precisely at resonance ($r = 0$) reduces equally the positive and negative parts of the profile. Thus the solution remains antit symmetric about $\bar{t} = \pi$, and the condition that F should have a zero mean value is satisfied without a shift in the position of the discontinuity. This is not the case for the near-resonance solutions for $|r| < 1$. Consider, for example, the solutions for $r = 0{\cdot}5$ and $0{\cdot}9$ [figures 4(*a*) and (*b*)]. Since these are not antisymmetric about $\bar{t} = \pi$, the zero mean value of F is maintained by small shifts of the position of the discontinuity.

From the solutions of both figures 4 and 5, we see that as δ is increased for a fixed value of Bu and r, a value δ_{crit} is reached at which the discontinuity in F disappears. (The solution for $r = 1$ [figure 4(*c*)] is the exception since the discontinuity has already just vanished at $\delta = 0$.) As δ is increased beyond δ_{crit}, the amplitude of F diminishes and the solution rapidly approaches the sinusoidal form predicted by the linear theory.

Although no solutions for $|r| > 1$ are presented, one would expect them to be qualitatively similar to those for $r = 1$, except that the profiles would all be smooth and generally less peaked at small values of δ.

The dependence of the magnitude of the discontinuity ΔF on r and δ for $Bu = 1$ is shown in figure 5. Although this figure is for the neighbourhood of the first resonant frequency, similar figures would be obtained for higher resonant frequencies. The locus of points in the plane $\Delta F = 0$ gives δ_{crit} as a function of r; it thus separates the regions in the r, δ plane in which shock waves do or do not exist. In the limit $\delta \to 0$ it can be shown from (58) that the dependence of ΔF on r is given by

$$\Delta F = 2(1-r^2)^{\frac{1}{2}}.$$

This relationship is displayed by the curve in the plane $\delta = 0$. Increasing δ from zero for a fixed value of r results in a nearly linear reduction of ΔF until the curve for δ_{crit} is approached.

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